

Planar flows and Plücker's type quadratic relations over semirings

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Abstract. It is well known, due to Lindström, that the minors of a (real or complex) matrix can be expressed in terms of weights of flows in a planar directed graph. Another classical fact is that there are plenty of homogeneous quadratic relations involving flag minors, or Plücker coordinates of the corresponding flag manifold. Generalizing and unifying these facts and their tropical counterparts, we consider a wide class of functions on $2^{[n]}$ that are generated by flows in a planar graph and take values in an arbitrary commutative semiring, where $[n] = \{1, 2, \dots, n\}$. We show that the “universal” homogeneous quadratic relations fulfilled by such functions can be described in terms of certain matchings, and as a consequence, give combinatorial necessary and sufficient conditions on the collections of subsets of $[n]$ determining these relations.

Keywords: Plücker relations, semiring, Laurent phenomenon, planar graph, network flow

AMS Subject Classification 05C75, 05E99

1 Introduction

For a positive integer n , let $[n]$ denote the set of integers $1, 2, \dots, n$.

In this paper we consider functions on the set $2^{[n]}$ of subsets of $[n]$ (or the n -dimensional Boolean cube) that take values in a commutative semiring and are generated by planar flows. Functions of this sort satisfy plenty of quadratic relations of Plücker's type, and our goal is to describe a combinatorial method that enables us to reveal and easily prove such relations.

We start with recalling some basic facts concerning Plücker algebra and Plücker coordinates. Consider the $n \times n$ matrix \mathbf{x} of indeterminates x_{ij} and its associated polynomial ring $\mathbb{Z}[\mathbf{x}]$. Also consider the polynomial ring $\mathbb{Z}[\Delta]$ associated to the set of 2^n variables Δ_S indexed by the subsets $S \subseteq [n]$. They are linked by the natural ring homomorphism $\psi : \mathbb{Z}[\Delta] \rightarrow \mathbb{Z}[\mathbf{x}]$ that brings each variable Δ_S to the flag minor polynomial for S , i.e. to the determinant of the submatrix \mathbf{x}_S formed by the column set S and the row set $\{1, \dots, |S|\}$ of \mathbf{x} . An important fact is that the ideal $\ker(\psi)$ of $\mathbb{Z}[\Delta]$ is generated by some homogeneous quadrics, each being an integer combination of products $\Delta_S \Delta_{S'}$ with the same parameter $(|S|, |S'|)$. They correspond to quadratic relations on the Plücker coordinates of a (real say) invertible $n \times n$ matrix (viz. on the

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Plücker coordinates of a point of the flag manifold over \mathbb{R}^n embedded in the appropriate projective space); for a survey see, e.g., [10, Ch. 14].

There are plenty of quadratic Plücker relations on flag minors of a matrix whose entries are assumed to belong to an arbitrary commutative ring \mathfrak{R} (of which the case $\mathfrak{R} = \mathbb{R}$ or \mathbb{C} is most popular). Let $f(S)$ denote the flag minor with a column set S in this matrix.

A well-known (and the simplest) special case of Plücker relations involves triples: for any three elements $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$, the flag minor function $f : 2^{[n]} \rightarrow \mathfrak{R}$ of an $n \times n$ matrix satisfies

$$f(Xik)f(Xj) - f(Xij)f(Xk) - f(Xjk)f(Xi) = 0, \quad (1.1)$$

where for brevity we write $Xi' \dots j'$ for $X \cup \{i', \dots, j'\}$. We refer to (1.1) as the *AP3-relation* (abbreviating “algebraic Plücker relation with triples”). Another well-known special case (in particular, encountered in a characterization of Grassmannians) involves quadruples $i < j < k < \ell$ and is viewed as

$$f(Xik)f(Xj\ell) - f(Xij)f(Xk\ell) - f(Xi\ell)f(Xjk) = 0. \quad (1.2)$$

A general (algebraic quadratic) Plücker relation on flag minors of a matrix can be written in the form

$$\sum_{A \in \mathcal{A}} f(X \cup \gamma_Y(A))f(X \cup \gamma_Y(\bar{A})) - \sum_{A' \in \mathcal{A}'} f(X \cup \gamma_Y(A'))f(X \cup \gamma_Y(\bar{A}')) = 0. \quad (1.3)$$

Here: (a) \mathcal{A} and \mathcal{A}' are certain collections of p -element subsets in $[p+q]$ for some integers $p, q > 0$ with $p+q \leq n$; (b) Y is a $(p+q)$ -element subset in $[n]$, and $\gamma = \gamma_Y$ is the order preserving bijective map $[p+q] \rightarrow Y$ (i.e. $\gamma(i) < \gamma(j)$ for $i < j$); (c) X is an arbitrary subset of $[n] - Y$; and (d) \bar{A} stands for the complement $[p+q] - A$ of $A \subseteq [p+q]$. Emphasize that (1.3) should be valid for the flag minor function f of any $n \times n$ matrix (over any \mathfrak{R}) and depends only on $p, q, \mathcal{A}, \mathcal{A}'$ but not X, Y . Note also that each of $\mathcal{A}, \mathcal{A}'$ is admitted to be a collection in which multiple sets $A \subseteq [p+q]$ are allowed (sometimes called a *multicollection*); in spite of this, to simplify notation we will write $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$.

In particular, (1.3) turns into (1.1) when $p = 2, q = 1, \mathcal{A} = \{13\}, \mathcal{A}' = \{12, 23\}$ and $Y = \{i, j, k\}$, and turns into (1.2) when $p = q = 2, \mathcal{A} = \{13\}, \mathcal{A}' = \{12, 14\}$ and $Y = \{i, j, k, \ell\}$.

An important property shown by Lindström [9] is that the minors of a matrix can be expressed by use of flows in a planar graph. A flow model will be the focus of our further description, and we now specify the notion of planar flows that we deal with. (See also [11] for further applications of the flow model.)

By a *planar network* we mean a finite directed planar graph $G = (V, E)$ (properly embedded in the plane) in which two n -element subsets $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_n\}$ of vertices are distinguished, called the sets of *sources* and *sinks* in G , respectively. We throughout assume that (a) G is (weakly) connected, that (b) the sources and sinks belong to the boundary (of the outer face) of G and occur in it in

the cyclic order $s_n, \dots, s_1, t_1, \dots, t_n$ (with possibly $s_1 = t_1$ or $s_n = t_n$), and that (c) G is *acyclic*, i.e. contains no directed cycle. We will attribute the term “network” to the graph G itself when the sets of sources and sinks in it are clear from the context. An important particular case is the *half-grid* Γ_n whose vertices are the integer points $(i, j) \in \mathbb{R}^2$ with $1 \leq j \leq i \leq n$, the edges are all possible ordered pairs of the form $((i, j), (i-1, j))$ or $((i, j), (i, j+1))$, the sources are $s_i = (i, 1)$ and the sinks are $t_i = (i, i)$, $i = 1, \dots, n$. The half-grid Γ_5 is illustrated in Fig. 1.

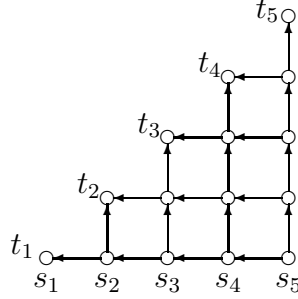


Figure 1: The half-grid Γ_5

For $I \subseteq [n]$, define $S_I := \{s_i : i \in I\}$ and $T_I := \{t_i : i \in I\}$. Speaking of an (I, J) -flow, where $I, J \subseteq [n]$ and $|I| = |J| =: k$, we mean a collection ϕ of k pairwise (vertex) disjoint directed paths in G going from the source set S_I to the sink set T_J . When ϕ enters the first k sinks (i.e. $J = [k]$), we refer to ϕ as a *flag flow* for I , or an I -flow. The set of I -flows (resp. (I, J) -flows) in G is denoted by $\Phi_I = \Phi_I^G$ (resp. $\Phi_{I,J} = \Phi_{I,J}^G$).

Let $w : V \rightarrow \mathfrak{R}$ be a weighting on the vertices of G , where, as before, \mathfrak{R} is a commutative ring. We associate to w the function $f = f_w$ on $2^{[n]}$ defined by

$$f(I) := \sum_{\phi \in \Phi_I} \prod_{v \in V_\phi} w(v), \quad I \subseteq [n], \quad (1.4)$$

where V_ϕ is the set of vertices occurring in a flow ϕ . (It is possible that G has no flag flow for some I , in which case $f(I)$ becomes 0.) We refer to f obtained in this way as an *algebraic flow-generated function*, or an *AFG-function* for short. By Lindström theorem [9], if M is the $n \times n$ matrix whose entries m_{ji} are defined as $\sum_{\phi \in \Phi_{\{i\}, \{j\}}} \prod_{v \in V_\phi} w(v)$, then for any $I, J \subseteq [n]$ with $|I| = |J|$, the minor of M with the column set I and the row set J is equal to $f(I, J)$, where the latter is defined as in expression (1.4) with Φ_I replaced by $\Phi_{I,J}$. A converse property takes place as well (at least for $\mathfrak{R} = \mathbb{R}$ or \mathbb{C}): the minors of any $n \times n$ matrix can be expressed as above via flows for some planar network and weighting.

Another important application of the flow model concerns tropical analogues of the above quadratic relations. In this case the flow-generated function $f = f_w$ on $2^{[n]}$ determined by a weighting w on V is defined as

$$f(I) := \max_{\phi \in \Phi_I} \left(\sum_{v \in V_\phi} w(v) \right), \quad I \subseteq [n]. \quad (1.5)$$

Here w is assumed to take values in a totally ordered abelian group \mathfrak{L} (usually one deals with $\mathfrak{L} = \mathbb{R}$ or \mathbb{Z}). The formula for f in (1.5) is nothing else than the tropicalization of

that in (1.4), and f is said to be a *tropical flow-generated function*, or a *TFG-function*. Some appealing properties of such functions and related objects are demonstrated in [3]. (See also [4] for additional results. Note that [3, 4] deal with real-valued tropical functions but everywhere \mathbb{R} can be replaced by \mathfrak{L} .) In particular, one shows that a TFG-function f satisfies the tropical analog of (1.1), or the *TP3-relation*:

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk), f(Xjk) + f(Xi)\}, \quad (1.6)$$

where, as before, $i < j < k$ and $X \subseteq [n] - \{i, j, k\}$. It turns out that a converse property holds as well: any function $f : 2^{[n]} \rightarrow \mathfrak{L}$ obeying the TP3-relation (for all i, j, k, X) is a TFG-function (determined by some G and w). In fact, to generate the set $\mathbf{T}_n(\mathfrak{L})$ of all TFG-functions it suffices to consider only one planar network, namely, the above-mentioned half-grid $\Gamma_n = (V, E)$, which emphasizes an important role of the latter. More precisely, the correspondence $w \mapsto f_w$, where $w : V \rightarrow \mathfrak{L}$, gives a bijection between \mathfrak{L}^V and $\mathbf{T}_n(\mathfrak{L})$. Two more results shown in [3] by handling flows in Γ_n are:

(i) $\mathbf{T}_n(\mathfrak{L})$ has as a sort of *basis* the set \mathcal{I}_n of all intervals $[p..q] := \{p, p+1, \dots, q\}$ in $[n]$ (including the “empty interval” \emptyset). This means that the restriction map $f \mapsto f|_{\mathcal{I}_n}$ gives a bijection between $\mathbf{T}_n(\mathfrak{L})$ and $\mathfrak{L}^{\mathcal{I}_n}$ (i.e. any TFG-function is determined by its values on the intervals, and those values can be chosen arbitrarily in \mathfrak{L});

(ii) for any subset $X \subseteq [n]$, the value of a TFG-function f on X can be expressed by a tropical Laurent polynomial in variables $f(I)$, $I \in \mathcal{I}_n$.

(Note that in general $\mathbf{T}_n(\mathfrak{L})$ admits many bases as in (i); an especial role of the basis \mathcal{I}_n is discussed in [3] where this basis is called *standard*. Note also that (ii) gives a tropical analogue of the Laurentness phenomenon for algebraic flow-generated functions f , i.e. the values of f are Laurent polynomials in the values on intervals, in the assumption that the latter ones are positive; see [5].)

In this paper we combine both algebraic and tropical cases by considering functions taking values in an arbitrary *commutative semiring* \mathfrak{S} , a set equipped with two associative and commutative binary operations \oplus (addition) and \odot (multiplication) satisfying the distributive law $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$. Sometimes we assume, in addition, that \mathfrak{S} contains neutral elements $\underline{0}$ (for addition) and/or $\underline{1}$ (for multiplication). Two special cases are of especial interest for us. When $\underline{0} \in \mathfrak{S}$ and each element has an additive inverse, \mathfrak{S} becomes a commutative ring as above. Another case is a commutative semiring with division, i.e. $\underline{1} \in \mathfrak{S}$ and each element has a multiplicative inverse. Examples of the latter are: the set $\mathbb{R}_{>0}$ of positive reals (with $\oplus = +$ and $\odot = \cdot$), and the above-mentioned tropicalization of a totally ordered abelian group \mathfrak{L} , denoted as $\mathfrak{L}^{\text{trop}}$ (with $\oplus = \max$ and $\odot = +$).

Extending (1.4) and (1.5), we define the flow-generated function $f = f_w$ determined by a weighting $w : V \rightarrow \mathfrak{S}$ as

$$f(I) := \bigoplus_{\phi \in \Phi_I} w(\phi), \quad I \subseteq [n], \quad (1.7)$$

where $w(\phi)$ stands for the weight $\odot(w(v) : v \in V_\phi)$ of a flow ϕ . We call f an *SFG-function* (abbreviating “flow-generated function over a semiring”), and denote the set of these functions by $\mathbf{FG} = \mathbf{FG}_n(\mathfrak{S})$. A direct analogue of identity (1.3) for \mathfrak{S} is

viewed as

$$\begin{aligned} \bigoplus_{A \in \mathcal{A}} (f(X \cup \gamma_Y(A)) \odot f(X \cup \gamma_Y(\bar{A}))) \\ = \bigoplus_{A' \in \mathcal{A}'} (f(X \cup \gamma_Y(A')) \odot f(X \cup \gamma_Y(\bar{A}'))), \end{aligned} \quad (1.8)$$

and when this holds true for fixed (multi)collections $\mathcal{A}, \mathcal{A}'$ and for any corresponding \mathfrak{S}, G, w, X, Y , we say that (1.8) is a *stable quadratic relation*, or an *sq-relation*.

Remark 1. If G and I are such that $\Phi_I^G = \emptyset$, then (1.7) is not applicable in general. In this case $f(I)$ may be regarded as *undefined*, and whenever expression (1.8) contains a summand $f(I) \odot f(J)$ with at least one of $f(I), f(J)$ being undefined (for the given G), we may think that this summand simply vanishes in the expression. (An alternative way is to put $f(I)$ to be an “extra neutral” element $*$ added to \mathfrak{S} , setting $* \oplus a = a$ and $* \odot a = *$ for all $a \in \mathfrak{S}$.) In particular, $f(\emptyset)$ may be regarded as undefined, and we will usually ignore the value of an SFG-function on the element \emptyset .

The goal of this paper is to describe a relatively simple combinatorial method of constructing pairs $\mathcal{A}, \mathcal{A}'$ determining sq-relations, and we give necessary and sufficient conditions on such pairs. In fact, our method is inspired by flow rearranging techniques elaborated in [3] for proving the TP3-relation for TFG-functions. The method reduces the task to a combinatorial problem of smaller size (and provides a polynomial-time algorithm to recognize whether or not a pair $\mathcal{A}, \mathcal{A}'$ gives an sq-relation). This is exposed in Theorem 3.1 which bridges validity of (1.8) for $p, q, \mathcal{A}, \mathcal{A}'$ and the property that two collections of certain *matchings* associated to $\mathcal{A}, \mathcal{A}'$ are *balanced*; the meaning of the latter notion will be explained later. It should be noted that our method of handling flows resembles, to some extent, a technique in [8] where quadratic relations on the amounts of perfect matchings in certain subgraphs of a planar graph are established.

The paper is organized as follows. Section 2 describes properties of certain pairs of flows (*double flows*) which lie in the background of our method. Section 3 states the main result (Theorem 3.1) and proves the sufficiency part in it, claiming that all balanced collections $\mathcal{A}, \mathcal{A}'$ generate sq-relations. Section 4 is devoted to illustrations of the method, which demonstrate a number of particular and wider classes of stable identities (1.8). Section 5 proves the necessity part in the main theorem; moreover, we show that if collections $\mathcal{A}, \mathcal{A}'$ are not balanced, then the corresponding quadratic relation does not hold already for some AFG-function with $\mathfrak{R} = \mathbb{R}$. This implies that *for $\mathcal{A}, \mathcal{A}'$ fixed, validity of (1.8) for all \mathfrak{S} is equivalent to validity of (1.3) for \mathbb{R}* . (This responds the so-called *transfer principle* for semirings; see, e.g., [1, Sec. 3].) The final Section 6 contains a short discussion on the standard basis and the Laurent phenomenon for SFG-functions over a commutative semiring with division.

2 Flows and double flows

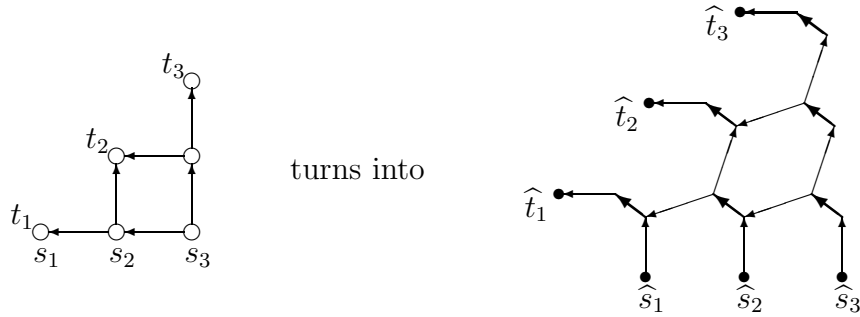
Let $G = (V, E)$ be a planar network with sources s_1, \dots, s_n and sinks t_1, \dots, t_n arranged as above, and let $p, q \in \mathbb{N}$ and $p + q \leq n$. As before, we assume that G is (weakly) connected and acyclic. In this section we describe ideas and tools behind the method

of constructing (multi)collections $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$ that ensure validity of (1.8) for all flow-generated functions $f = f_w$ on $2^{[n]}$ determined by weightings $w : V \rightarrow \mathfrak{S}$, where \mathfrak{S} is an arbitrary commutative semiring.

First of all we specify some terminology and notation. By a *path* in a digraph (directed graph) we mean a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ where each e_i is an edge connecting vertices v_{i-1}, v_i . An edge e_i is called *forward* if it is directed from v_{i-1} to v_i , denoted as $e_i = (v_{i-1}, v_i)$, and *backward* otherwise (when $e_i = (v_i, v_{i-1})$). The path P is called *directed* if it has only forward edges, and *simple* if all vertices v_i are distinct. When $k > 0$, $v_0 = v_k$ and all v_1, \dots, v_k are distinct, P is called a *simple cycle*, or a *circuit*; it is often considered up to cyclically shifting and reversing. The sets of vertices and edges of P are denoted by V_P and E_P , respectively.

Recall that by an I -flow in G , where $I \subseteq [n]$, we mean a collection ϕ of $|I|$ pairwise disjoint directed paths going from the source set S_I to the sink set $\{t_1, \dots, t_{|I|}\}$. Since G is acyclic, all these paths are simple, and the order of sources and sinks in the boundary of G implies that the path in ϕ entering a sink t_i begins at i -th source in S_I (in the natural ordering there). A useful equivalent definition of an I -flow ϕ is that $\delta_\phi^{\text{out}}(s_i) = 1$ and $\delta_\phi^{\text{in}}(s_i) = 0$ if $i \in I$; $\delta_\phi^{\text{out}}(t_j) = 0$ and $\delta_\phi^{\text{in}}(t_j) = 1$ if $1 \leq j \leq |I|$; and $\delta_\phi^{\text{out}}(v) = \delta_\phi^{\text{in}}(v) \in \{0, 1\}$ for the other vertices v in G . Here $\delta_\phi^{\text{out}}(v)$ (resp. $\delta_\phi^{\text{in}}(v)$) denotes the number of edges in ϕ leaving (resp. entering) a vertex v . Also we denote $\delta_\phi^{\text{out}}(v) + \delta_\phi^{\text{in}}(v)$ by $\delta_\phi(v)$.

Our approach is based on examining certain pairs of flag flows in G and rearranging them to form some other pairs. To simplify technical details, it is convenient to consider an equivalent flow model, obtained by slightly modifying the network G , as follows. Let us split each vertex $v \in V$ into two vertices v', v'' (disposing them in a small neighborhood of v in the plane) and connect them by edge $e_v = (v', v'')$, called a *split-edge*. Each edge (u, v) of G is replaced by an edge going from u'' to v' ; we call it an *ordinary* edge. Also for each $s_i \in S$, we add new source \hat{s}_i and edge (\hat{s}_i, s'_i) , and for each $t_j \in T$, add new sink \hat{t}_j and edge (t''_j, \hat{t}_j) ; we refer to such edges as *extra* ones. The picture illustrates the transformation for $G = \Gamma_3$.



Note that the new (modified) network is again acyclic, but it need not be planar in general. Nevertheless, we keep the same notation $G = (V, E)$ for it, and take $\hat{S} := \{\hat{s}_1, \dots, \hat{s}_n\}$ and $\hat{T} := \{\hat{t}_1, \dots, \hat{t}_n\}$ as the sets of sources and sinks in it, respectively. Sources and sinks are also called *terminal* vertices. Clearly for any $i, j \in [n]$, there is a natural 1–1 correspondence between the directed paths from s_i to t_j in the initial network and the ones from \hat{s}_i to \hat{t}_j in the modified network. This is extended to a

natural 1–1 correspondence between flag flows, and for $I \subseteq [n]$, we keep notation Φ_I for the set of flows going from $\widehat{S}_I := \{\widehat{s}_i : i \in I\}$ to $\widehat{T}_{[I]} := \{\widehat{t}_1, \dots, \widehat{t}_{|I|}\}$. A weighting w on the vertices v of the initial G is transferred to the split-edges of the modified G , namely, $w(e_v) := w(v)$. Then corresponding flows in both networks have equal weights (which are the products by \odot of the weights of vertices or split-edges in the flows). This implies that the functions on $2^{[n]}$ generated by corresponding flows coincide.

We will take advantages from the following obvious property of the modified G :

- (2.1) each non-terminal vertex u is incident with exactly one split-edge e , and if e enters (leaves) u , then $\delta_G^{\text{in}}(u) = 1$ (resp. $\delta_G^{\text{out}}(u) = 1$); each terminal vertex has exactly one incident edge.

Let X, Y be disjoint subsets of $[n]$ and $|Y| = p + q$. We assume that $p \geq q$ and, as before, denote by $\gamma = \gamma_Y$ the order preserving bijective map of $[p + q]$ to Y . We write $C \Delta D$ for the symmetric difference $(C - D) \cup (D - C)$ of subsets C, D of a set.

Let us fix a subset $A \in \binom{[p+q]}{p}$. Define $I(A) := X \cup \gamma(A)$ and $J(A) := X \cup \gamma(\overline{A})$, where $\overline{A} := [p + q] - A$. Consider an $I(A)$ -flow ϕ and a $J(A)$ -flow ϕ' in G . Our method will rely on the following three lemmas.

Lemma 2.1 *$E_\phi \Delta E_{\phi'}$ is partitioned into the edge sets of pairwise disjoint circuits C_1, \dots, C_d (for some d) and simple paths P_1, \dots, P_p , where each P_i connects a source in $\widehat{S}_{\gamma(A)}$ with either a source in $\widehat{S}_{\gamma(\overline{A})}$ or a sink in the set $\widetilde{T} := \widehat{T}_{|X|+p} - \widehat{T}_{|X|+q}$. In each of these circuits and paths, the edges of ϕ and the edges of ϕ' have opposed directions (say, the former edges are forward and the latter ones are backward).*

Proof Observe that a vertex v of G satisfies: (i) $\delta_\phi(v) = 1$ and $\delta_{\phi'}(v) = 0$ if $v \in \widehat{S}_{\gamma(A)} \cup \widetilde{T}$; (ii) $\delta_\phi(v) = 0$ and $\delta_{\phi'}(v) = 1$ if $v \in \widehat{S}_{\gamma(\overline{A})}$; (iii) $\delta_\phi(v) = \delta_{\phi'}(v) = 1$ if $v \in \widehat{S}_X \cup \widehat{T}_{|X|+q}$; and (iv) $\delta_\phi(v), \delta_{\phi'}(v) \in \{0, 2\}$ otherwise. This together with property (2.1) implies that any vertex v is incident with 0, 1 or 2 edges in $E_\phi \Delta E_{\phi'}$, and the number is equal to 1 if and only if $v \in \widehat{S}_{\gamma(A)} \cup \widehat{S}_{\gamma(\overline{A})} \cup \widetilde{T}$. Hence the weakly connected components of the subgraph of G induced by $E_\phi \Delta E_{\phi'}$ are circuits, C_1, \dots, C_d say, and simple paths P_1, \dots, P_p , each of the latter connecting two vertices in $\widehat{S}_{\gamma(A)} \cup \widehat{S}_{\gamma(\overline{A})} \cup \widetilde{T}$.

Consider consecutive edges e, e' in a circuit C_i or a path P_j . If both e, e' belong to the same flow among ϕ, ϕ' , then, obviously, they have the same direction in this circuit/path. Suppose e, e' belong to different flows. In view of (2.1), the common vertex v of e, e' is non-terminal and incident with a split-edge e'' . Clearly e'' belongs to both ϕ, ϕ' , and therefore $e'' \neq e, e'$. This implies that either both e, e' enter v or both leave v , so they are directed differently along the circuit/path containing them. This yields the second assertion in the lemma.

Finally, suppose some path P_j has both ends in $\widehat{S}_{\gamma(A)}$. Then the first and last edges e, e' of P_j are extra edges of G contained in ϕ . But both e, e' leave $\widehat{S}_{\gamma(A)}$, so they are directed differently along P_j , contrary to proved above. Thus, each path P_j has exactly one end in $\widehat{S}_{\gamma(A)}$ (in view of $|\widehat{S}_{\gamma(A)}| = |\widehat{S}_{\gamma(\overline{A})}| + |\widetilde{T}|$), completing the proof. \blacksquare

Figure 2 illustrates an example of G, ϕ, ϕ', ξ , and $E_\phi \Delta E_{\phi'}$.

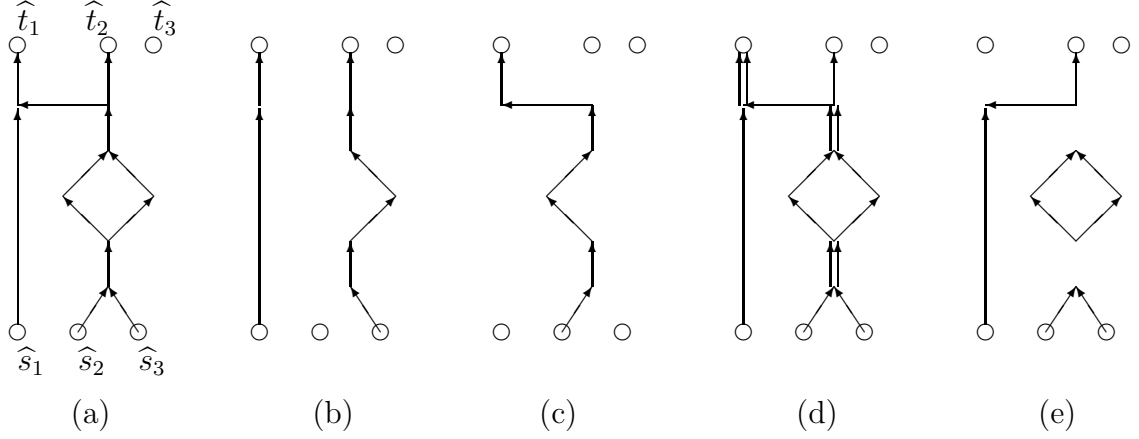


Figure 2: (a) G ; (b) ϕ ; (c) ϕ' ; (d) ξ ; (e) $E_\phi \Delta E_{\phi'}$

Let $\chi^{E'}$ denote the incidence vector in \mathbb{R}^E of a subset E' of edges of G , i.e. $\chi^{E'}(e) = 1$ if $e \in E'$, and 0 otherwise. A function $\xi : E \rightarrow \{0, 1, 2\}$ is called a *double flow* for the sets $I(A), J(A)$ as above if there exist an $I(A)$ -flow ϕ and a $J(A)$ -flow ϕ' such that $\chi^{E_\phi} + \chi^{E_{\phi'}} = \xi$. We say that ϕ, ϕ' *decompose* ξ and denote the number of such pairs ϕ, ϕ' by $N_{I(A), J(A)}(\xi)$. For $i = 1, 2$, let $\xi^{(i)}$ denote the set of edges e with $\xi(e) = i$, and let $d(\xi)$ denote the number of circuits in the subgraph of G induced by $\xi^{(1)}$ (i.e. the number d in Lemma 2.1).

Lemma 2.2 $N_{I(A), J(A)}(\xi) = 2^{d(\xi)}$.

Proof Fix a pair $\phi \in \Phi_{I(A)}$ and $\phi' \in \Phi_{J(A)}$ decomposing ξ . Let $C_1, \dots, C_{d(\xi)}$ be the circuits in the subgraph induced by $\xi^{(1)}$ ($= E_\phi \Delta E_{\phi'}$). By Lemma 2.1, each circuit C_i is a concatenation of (directed up to reversing) paths $Q_1, \dots, Q_{2k} = Q_0$, where consecutive Q_j, Q_{j+1} are contained in different flows among ϕ, ϕ' and either both leave or both enter their common vertex (note that C_i cannot entirely belong to one of ϕ, ϕ' since G is acyclic). Therefore, exchanging the pieces Q_j in ϕ, ϕ' (i.e. replacing E_ϕ by $E_\phi \Delta E_{C_i}$, and $E_{\phi'}$ by $E_{\phi'} \Delta E_{C_i}$), we obtain a decomposition of ξ into another pair of $I(A)$ - and $J(A)$ -flows.

The above procedure can be applied to circuits C_i independently. So we can choose an arbitrary subset $D \subseteq [d(\xi)]$. Let $\mathcal{D} := \cup(E_{C_i} : i \in D)$. Then the edge set $E_\phi \Delta \mathcal{D}$ induces an $I(A)$ -flow ψ , and $E_{\phi'} \Delta \mathcal{D}$ induces a $J(A)$ -flow ψ' . Furthermore, $\chi^{E_\psi} + \chi^{E_{\psi'}} = \xi$. Obviously, different subsets D produce different pairs ψ, ψ' decomposing ξ (e.g., the choice $D = \emptyset$ gives the initial pair ϕ, ϕ'). Conversely, if a pair of $\psi \in \Phi_{I(A)}$ and $\psi' \in \Phi_{J(A)}$ decomposes ξ , then, in view of $E_\psi \cap E_{\psi'} = \xi^{(2)}$ and $E_\psi \Delta E_{\psi'} = \xi^{(1)}$, one can conclude that for each circuit C_i , the components of $C_i \cap \psi$ and $C_i \cap \psi'$ are the alternating subpaths Q_j as above. This gives the desired equality. \blacksquare

Next we are going to decompose ξ into a pair of flows whose source sets are different from $\widehat{S}_{\gamma(A)}, \widehat{S}_{\gamma(\overline{A})}$. Let P_1, \dots, P_p be the paths as in Lemma 2.1. Exactly q of them connect $\widehat{S}_{\gamma(A)}$ and $\widehat{S}_{\gamma(\overline{A})}$; we call these paths *essential* and denote their set by $\mathcal{P}(\xi)$. For a path $P \in \mathcal{P}(\xi)$ with end vertices $\widehat{s}_{\gamma(i)}$ and $\widehat{s}_{\gamma(j)}$, the pair $\{i, j\}$ is denoted by $\pi(P)$ (in particular, $|\pi(P) \cap A| = |\pi(P) \cap \overline{A}| = 1$). Define $M(\xi) := \{\pi(P) : P \in \mathcal{P}(\xi)\}$.

Lemma 2.3 Choose an arbitrary subset $\Pi \subseteq M(\xi)$. Define $Z := \cup(\pi \in \Pi)$ and $A' := A \triangle Z$. Then ξ is decomposed by exactly $2^{d(\xi)}$ pairs formed by an $I(A')$ -flow and a $J(A')$ -flow. Therefore, $N_{I(A'), J(A')}(\xi) = N_{I(A), J(A)}(\xi)$.

Proof Each path $P \in \mathcal{P}(\xi)$ is a concatenation of an even number of subpaths Q_1, \dots, Q_r alternately contained in ϕ and ϕ' . Let $\pi(P) = \{i, j\}$; then one of i, j is in A , and the other in \bar{A} . Also one source among $\widehat{s}_{\gamma(i)}, \widehat{s}_{\gamma(j)}$ belongs to Q_1 , and the other to Q_r . Exchanging in ϕ, ϕ' the corresponding pieces Q_k of P , we obtain a decomposition of ξ into an $I(\bar{A})$ -flow and a $J(\bar{A})$ -flow, where $\bar{A} := A \triangle \{i, j\}$. Now the result is obtained by arguing as in Lemma 2.2. \blacksquare

In what follows we will use the fact that, although the modified graph G may not be planar, its subgraph $G(\xi)$ induced by the edge set $\xi^{(1)} \cup \xi^{(2)}$ is planar.

To see this, let ξ be decomposed by flows ϕ, ϕ' (as before) and consider in the initial graph G a non-terminal vertex v which belongs to both flows ϕ, ϕ' . Let a, a' be the edges of ϕ entering and leaving v , respectively, and let b, b' be similar edges for ϕ' . The only situation when the modified graph G is not locally planar in a small neighborhood of the split-edge e_v is that all a, a', b, b' are different and follow in this order (clockwise or counterclockwise) around v . We assert that this is not the case. Indeed, a, a' belong to a directed path P in G from a source s_i to a sink $t_{i'}$, and similarly there is a directed path Q from s_j to $t_{j'}$ containing b, b' . From the facts that the initial graph G is planar and acyclic and that the edges a, a', b, b' occur in this order around v one can conclude that the paths P, Q can meet only at v . This implies that the terminals $s_i, t_{i'}, s_j, t_{j'}$ are different and follow in this order in the boundary of G , yielding a contradiction. Thus, $G(\xi)$ is planar, as required.

3 Balanced collections and the main theorem

In this section we use the above observations and results to construct collections providing stable quadratic relations.

As before, consider a set $A \in \binom{[p+q]}{p}$, a double flow ξ for $I(A), J(A)$ and the set of pairs $M = M(\xi)$. It will be convenient to think that all pairs are ordered: if a pair π consists of elements i, j and $i < j$, we write $\pi = (i, j)$ or $\pi = ij$ and call it an *arc* in M . We denote the interval $\{i, i+1, \dots, j\}$ by $[i..j]$ or by $[\pi]$ and say that an element k is *covered* by π if $k \in [\pi]$. An element in $[p+q] - \cup(\pi \in M)$ is called *free*.

We observe that M possesses the following properties:

- (3.1) (i) $|M| = q$, the arcs in M are mutually disjoint, and $|\pi \cap A| = |\pi \cap \bar{A}| = 1$ for each $\pi \in M$;
- (ii) the set M is *nested*, which means that for any two arcs $\pi, \pi' \in M$, the intervals $[\pi]$ and $[\pi']$ are either disjoint or one includes the other;
- (iii) no free element is covered by an arc in M (i.e. $\pi \in M$ and $k \in [\pi]$ imply $k \in \pi'$ for some $\pi' \in M$).

Indeed, (i) is obvious. Violation of (ii) means the existence of arcs $\pi = ij$ and $\pi' = i'j'$ in M such that $i < i' < j < j'$. Then the sources $s_{\gamma(i)}, s_{\gamma(i')}, s_{\gamma(j)}, s_{\gamma(j')}$ follow in

this order in the boundary of the initial G . Since the graph $G(\xi)$ is planar, the path $P \in \mathcal{P}(\xi)$ connecting $\widehat{s}_{\gamma(i)}, \widehat{s}_{\gamma(j)}$ intersects the path $P' \in \mathcal{P}(\xi)$ connecting $\widehat{s}_{\gamma(i')}, \widehat{s}_{\gamma(j')}$. But P, P' must be disjoint (cf. Lemma 2.1). To see (iii), consider an arc $\pi = ij \in M$ and a free element k . Since k is free, the subgraph induced by $\xi^{(1)}$ contains a path P connecting the source $\widehat{s}_{\gamma(k)}$ and some sink $\widehat{t}_r \in \widetilde{T}$. In case $k \in [\pi]$, the path P would intersect the path in $\mathcal{P}(\xi)$ connecting $\widehat{s}_{\gamma(i)}$ and $\widehat{s}_{\gamma(j)}$ (since $G(\xi)$ is planar and $\widehat{s}_{\gamma(i)}, \widehat{s}_{\gamma(k)}, \widehat{s}_{\gamma(j)}, \widehat{t}_r$ follow in this order in its boundary), which is impossible.

The above observations inspire consideration of more abstract objects. A set M of ordered pairs (arcs) in $[p+q]$ satisfying (3.1) is called a *feasible matching* for A . The set of all feasible matchings for A is denoted by $\mathcal{M}(A)$, and we refer to a pair (A, M) , where $M \in \mathcal{M}(A)$, as a *configuration*. For a collection $\mathcal{A} \subseteq \binom{[p+q]}{p}$, the (multi)set of all configurations (A, M) with $A \in \mathcal{A}$ is denoted by $\mathcal{K}(\mathcal{A})$.

The *exchange operation* applied to a configuration (A, M) and to a chosen subset $\Pi \subseteq M$ makes the p -element set $A' := A \Delta (\cup(\pi \in \Pi))$; in other words, we swap the elements of A and \overline{A} in each arc $\pi \in \Pi$. Clearly M becomes a feasible matching for A' , and the exchange operation applied to the configuration (A', M) and the same subset Π returns A .

Definition. Let us say that two (multi)collections $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$ are *balanced* if there exists a bijection of $\mathcal{K}(\mathcal{A})$ to $\mathcal{K}(\mathcal{A}')$ that sends each configuration (A, M) in the former to a configuration (A', M) in the latter. (We rely on the simple fact that if (A, M) and (A', M) are two configurations with the same matching M , then A' can be obtained from A by the exchange operation w.r.t. some $\Pi \subseteq M$.) Equivalently, $\mathcal{A}, \mathcal{A}'$ are balanced if for each matching M in $[p+q]$, the number of times M occurs in sets $\mathcal{M}(A)$ among $A \in \mathcal{A}$ is equal to a similar number in sets $\mathcal{M}(A')$ among $A' \in \mathcal{A}'$. We can express this condition as

$$\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}'),$$

where for a collection $\mathcal{A}'' \subseteq \binom{[p+q]}{p}$, $\mathcal{M}(\mathcal{A}'')$ denotes the multiset consisting of matchings M taken with multiplicities $|\{A \in \mathcal{A}'' : M \in \mathcal{M}(A)\}|$.

This notion plays a central role in our description, and the main result is as follows.

Theorem 3.1 *Let $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$. The following statements are equivalent:*

- (i) *(1.8) is a stable quadratic relation;*
- (ii) *the pair $\mathcal{A}, \mathcal{A}'$ is balanced.*

Part (i) \Rightarrow (ii) of this theorem will be shown in Section 5. In its turn, part (ii) \Rightarrow (i) can be immediately proved by relying on the lemmas from the previous section.

Proposition 3.2 *Let $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$ be balanced. Then (1.8) holds for any disjoint subsets $X, Y \subseteq [n]$ with $|Y| = p+q$ and any SFG-function f on $2^{[n]}$ (concerning arbitrary G, w, \mathfrak{S} as above).*

Proof Fix corresponding G, w, \mathfrak{S}, X, Y and consider the function $f = f_w$ determined by the weighting w . For $A \in \binom{[p+q]}{p}$, let $\Xi(A)$ denote the set of all distinct double

flows ξ for $I(A), J(A)$ (considering G in the modified form). The summand concerning $A \in \mathcal{A}$ in the l.h.s. of (1.8) can be expressed via double flows as follows:

$$\begin{aligned} f(I(A)) \odot f(J(A)) &= \left(\bigoplus_{\phi \in \Phi_{I(A)}} w(\phi) \right) \odot \left(\bigoplus_{\phi' \in \Phi_{J(A)}} w(\phi') \right) \\ &= \bigoplus_{\xi \in \Xi(A)} N_{I(A), J(A)}(\xi) w^{\odot \xi}, \end{aligned} \quad (3.2)$$

where $N_{I(A), J(A)}(\xi)$ is the number of pairs $(\phi, \phi') \in (\Phi_{I(A)}, \Phi_{J(A)})$ with $\chi^{E_\phi} + \chi^{E_{\phi'}} = \xi$ (cf. Lemma 2.2), and $w^{\odot \xi}$ is the function on the set E_V of split-edges taking values $w(e) \odot \dots \odot w(e)$ ($\xi(e)$ times), $e \in E_V$. Do similarly for the summand concerning $A' \in \mathcal{A}'$ in the r.h.s. of (1.8).

We associate to each $\xi \in \Xi(A)$ the configuration $K_{A, \xi} := (A, M(\xi))$. Let $\beta : \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A}')$ be the corresponding bijection (existing as $\mathcal{A}, \mathcal{A}'$ are balanced). For $A \in \mathcal{A}$ and $\xi \in \Xi(A)$, β sends the configuration $K_{A, \xi}$ to a configuration (A', M) with $A' \in \mathcal{A}'$ and $M = M(\xi)$. Then ξ is a double flow for $I(A'), J(A')$ as well. Therefore, β gives a 1–1 correspondence between the set of pairs $(A \in \mathcal{A}, \xi \in \Xi(A))$ and the set of pairs $(A' \in \mathcal{A}', \xi' \in \Xi(A'))$. Moreover, corresponding pairs (A, ξ) and (A', ξ') are such that $\xi = \xi'$. By Lemma 2.3, we have $N_{I(A), J(A)}(\xi) = N_{I(A'), J(A')}(\xi)$. Now the desired equality (1.8) follows by comparing the \oplus -sum of the last terms in (3.2) over $A \in \mathcal{A}$ with a similar sum over $A' \in \mathcal{A}'$. \blacksquare

4 Examples of stable quadratic relations

In this section we illustrate the method described in the previous section by exhibiting several classes of stable quadratic relations on SFG-functions. According to Proposition 3.2, once we are able to show that one or another pair of collections $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$ is balanced, we can declare that relation (1.8) involving these collections is stable. Recall that speaking of a stable quadratic relation (with collections $\mathcal{A}, \mathcal{A}'$ fixed), or an *sq-relation* for short, we mean that (1.8) holds for any SFG-function $f : 2^{[n]} \rightarrow \mathfrak{S}$ (concerning arbitrary G, w, \mathfrak{S}) and any disjoint sets $X, Y \subseteq [n]$ with $|Y| = p + q$.

When considering and visualizing one or another ordered partition (A, \bar{A}) of $[p + q]$, it will be convenient for us to call elements of A *white*, and elements of \bar{A} *black*.

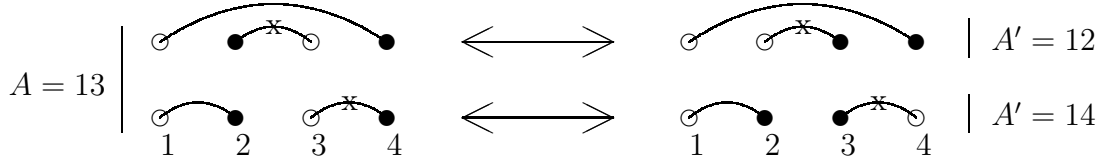
1. When $p = 2$ and $q = 1$, the collection $\binom{[p+q]}{p}$ consists of three 2-element sets A , namely, 12, 13, 23, and their complements \bar{A} are the 1-element sets 3, 2, 1, respectively. Since $q = 1$, a feasible matching consists of a unique arc. The sets 12 and 23 admit only one feasible matching each, namely, $\mathcal{M}(12) = \{\{23\}\}$ and $\mathcal{M}(23) = \{\{12\}\}$, whereas 13 has two feasible matchings, namely, $\mathcal{M}(13) = \{\{12\}, \{23\}\}$. Therefore, the collections $\mathcal{A} := \{13\}$ and $\mathcal{A}' := \{12, 23\}$ are balanced. The corresponding configurations and bijection are illustrated in the picture where the 2-element sets (forming $\mathcal{A}, \mathcal{A}'$) and their 1-element complements are indicated by white and black circles, respectively.



This gives rise to an sq-relation on triples in $[n]$ (generalizing AP3- and TP3-relations (1.1),(1.6)): for any $i < j < k$ (forming Y) and $X \subseteq [n] - \{i, j, k\}$, one holds:

$$f(Xik) \odot f(Xj) = (f(Xij) \odot f(Xk)) \oplus (f(Xjk) \odot f(Xi)). \quad (4.1)$$

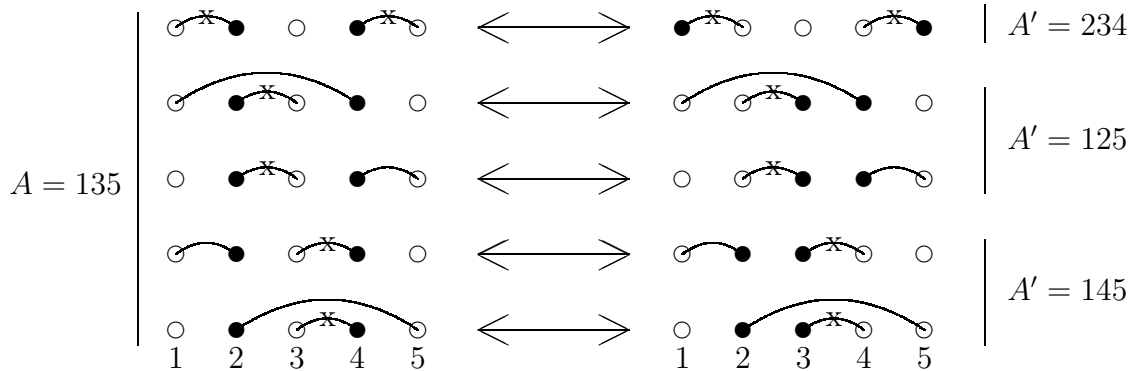
2. Let $p = q = 2$. Take the collections $\mathcal{A} := \{13\}$ and $\mathcal{A}' := \{12, 14\}$ in $\binom{[4]}{2}$. One can see that each of 12 and 14 admits a unique feasible matching: $\mathcal{M}(12) = \{14, 23\}$ and $\mathcal{M}(14) = \{12, 34\}$, whereas $\mathcal{M}(13)$ consists of two feasible matchings, just the same $\{14, 23\}$ and $\{12, 34\}$. Therefore, $\mathcal{A}, \mathcal{A}'$ are balanced; see the picture where the arcs involved in the corresponding exchange operations are marked with crosses.



As a consequence, we obtain an sq-relation on quadruples (generalizing (1.2) and its tropical counterpart): for any $i < j < k < \ell$ (forming Y) and $X \subseteq [n] - \{i, j, k, \ell\}$, one holds:

$$f(Xik) \odot f(Xj\ell) = (f(Xij) \odot f(Xk\ell)) \oplus (f(Xi\ell) \odot f(Xjk)). \quad (4.2)$$

3. As one more illustration of the method, let us consider one particular case for $p = 3$ and $q = 2$. Put $\mathcal{A} := \{135\}$ and $\mathcal{A}' := \{234, 125, 145\}$. One can check that $\mathcal{M}(234) = \{\{12, 45\}\}$, $\mathcal{M}(125) = \{\{14, 23\}, \{23, 45\}\}$, $\mathcal{M}(145) = \{\{12, 34\}, \{25, 34\}\}$, and that $\mathcal{M}(135)$ consists just of the five matchings occurring in those three collections. Therefore, $\mathcal{A}, \mathcal{A}'$ are balanced. The corresponding configurations and bijection are shown in the picture.



This implies a particular sq-relation on quintuples: for $i < j < k < \ell < m$ and $X \subseteq [n] - \{i, j, k, \ell, m\}$, one holds:

$$f(Xikm) \odot f(Xj\ell) = (f(Xjkl) \odot f(Xim)) \oplus (f(Xijm) \odot f(Xk\ell)) \oplus (f(Xilm) \odot f(Xjk)). \quad (4.3)$$

4. Next we describe a wide class of balanced collections for arbitrary $p \geq q$; it includes the collections indicated in items 1 and 2 as very special cases.

The collection \mathcal{A} that we are going to construct contains two distinguished sets B_0, B_1 . The set B_0 is the interval $[p]$. Then $\overline{B}_0 = [p+1..p+q]$ and $\mathcal{M}(B_0)$ consists of a unique matching M_0 : its arcs are the pairs $\pi_i := (p-i+1, p+i)$ for $i = 1, \dots, q$. The set B_1 is obtained by applying to B_0 the exchange operation w.r.t. a chosen nonempty subset $\Pi_0 \subseteq M_0$, i.e. $B_1 = L \cup R$, where

$$L := [p-q] \cup \{p-i+1 : \pi_i \notin \Pi_0\} \quad \text{and} \quad R := \{p+i : \pi_i \in \Pi_0\}. \quad (4.4)$$

An example for $p = 5, q = 4$ is drawn in the picture where the arcs in Π_0 are marked with crosses.



The other members of $\mathcal{A} \cup \mathcal{A}'$ have the same tail part R as the set B_1 . More precisely, take the collection

$$\mathcal{B} := \{A \subseteq [p+q] : |A| = p, A \cap [p+1..p+q] = R\}.$$

For a subset $A \subseteq [n]$, define $\Sigma(A) := \sum(i \in A)$. Now put

$$\begin{aligned} \mathcal{A} &:= \{B_0\} \cup \{A \in \mathcal{B} : \Sigma(A) - \Sigma(B_1) \text{ odd}\} \quad \text{and} \\ \mathcal{A}' &:= \{A \in \mathcal{B} : \Sigma(A) - \Sigma(B_1) \text{ even}\}. \end{aligned} \quad (4.5)$$

In particular, $B_0 \in \mathcal{A}$, $B_1 \in \mathcal{A}'$, and $\mathcal{A} \cap \mathcal{A}' = \emptyset$.

Lemma 4.1 *The pair $\mathcal{A}, \mathcal{A}'$ as in (4.5) is balanced.*

Proof Consider a set $A \in \mathcal{A} \cup \mathcal{A}'$ and a matching $M \in \mathcal{M}(A)$. We describe a rule which associates to (A, M) another configuration (A', M) (aiming to obtain a bijection between $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A}')$). Two cases are possible.

Case 1: $A = B_0$. Then $M = M_0$, and the configuration (B_0, M_0) belongs to $\mathcal{K}(\mathcal{A})$. We naturally associate to (B_0, M_0) the configuration (B_1, M_0) in $\mathcal{K}(\mathcal{A}')$. (Note that B_0 and B_1 are linked by the exchange operation w.r.t. the set $\Pi_0 \subseteq M_0$.)

Case 2: $A \neq B_0$. Then $A \in \mathcal{B}$. Suppose M consists of arcs $\rho_k = i_k j_k$, $k = 1, \dots, q$, and let $j_1 < j_2 < \dots < j_q$. Let us say that an arc ρ_k is *short* if $j_k = i_k + 1$. It

is immediate from (3.1) that the interval of any arc $\rho_{k'}$ contains a short arc ρ_k , i.e. $i_{k'} \leq i_k < j_k \leq j_{k'}$. This implies that the arc ρ_1 is short, in view of $j_1 < \dots < j_q$. Consider two subcases.

Subcase 2a: $j_1 \geq p + 1$. This is possible only if $j_k = p + k$ for all $k = 1, \dots, q$. Then $[\rho_1] \subset [\rho_2] \subset \dots \subset [\rho_q]$, implying $i_k = p + 1 - k$ (in view of (3.1)(ii),(iii)). Therefore, M coincides with M_0 , and now the condition $A \cap [p + 1..p + q] = R$ implies that A coincides with the set B_1 . So (A, M) is the configuration (B_1, M_0) in $\mathcal{K}(\mathcal{A})$, and we associate to it the configuration (B_0, M_0) , to be agreeable with Case 1.

Subcase 2b: $j_1 \leq p$. We associate to (A, M) the configuration (A', M) with $A' := A \triangle \rho_1$ (i.e. we apply to (A, M) the exchange operation w.r.t. the subset of M formed by the singleton $\{\rho_1\}$). Obviously, $A' \cap [p + 1..p + q] = R$, whence $A' \in \mathcal{B}$. Also $j_1 - i_1 = 1$ implies that $\Sigma(A) - \Sigma(A')$ is odd. Thus, one of $(A, M), (A', M)$ belongs to $\mathcal{K}(\mathcal{A})$ and the other to $\mathcal{K}(\mathcal{A}')$. We associate these configurations to each other (taking into account that the same short arc ρ_1 in M is chosen in both cases). \blacksquare

Remark 2. (i) When $p = 2$ and $q = 1$, we have $B_0 = 12$, $\overline{B}_0 = 3$ and $M_0 = \{23\}$. Taking $\Pi_0 = M_0$, we obtain $B_1 = 13$ and $\mathcal{B} = \{13, 23\}$. This gives $\mathcal{A} = \{12, 23\}$ and $\mathcal{A}' = \{13\}$, which matches the balanced collections described in item 1. (ii) When $p = q = 2$, we have $B_0 = 12$, $\overline{B}_0 = 34$ and $M_0 = \{23, 14\}$. Taking $\Pi_0 = \{23\}$, we obtain $B_1 = 13$ and $\mathcal{B} = \{13, 23\}$. This gives $\mathcal{A} = \{12, 23\}$ and $\mathcal{A}' = \{13\}$, which is equivalent to the balanced collections $\{13\}, \{12, 14\}$ in item 2.

Pairs $\mathcal{A}, \mathcal{A}'$ as in (4.5) give rise to sq-relations on SFG-functions which are viewed as follows. Let $Y \subseteq [n]$ consist of elements $i_1 < \dots < i_p < j_1 < \dots < j_q$ and let $X \subseteq [n] - Y$. Put $I := \{i_1, \dots, i_p\}$ and $J := \{j_1, \dots, j_q\}$ and choose a subset $R \subseteq J$ (which corresponds to R in (4.4)). Then the corresponding sq-relation is:

$$\begin{aligned} (f(X \cup I) \odot f(X \cup J)) \oplus \bigoplus_{I' \in \mathcal{I}^{\text{odd}}} f(X \cup (I - I') \cup R) \odot f(X \cup I' \cup (J - R)) \\ = \bigoplus_{I'' \in \mathcal{I}^{\text{even}}} f(X \cup (I - I'') \cup R) \odot f(X \cup I'' \cup (J - R)). \end{aligned} \quad (4.6)$$

Here the collection $\mathcal{I}^{\text{even}}$ (resp. \mathcal{I}^{odd}) is formed by the subsets $\tilde{I} \subseteq I$ such that $|\tilde{I}| = |R|$ and the integers $\sum(k: j_k \in R)$ and $\sum(p + 1 - k: i_k \in \tilde{I})$ have the same (resp. different) parity.

When $p = q$ and $\mathfrak{S} = \mathbb{C}$, relations similar to (4.6) appear in a characterization of the Grassmannian $G_{d,n}$. In this case one should take all possible tuples p, Y, X, R such that $2 \leq p \leq d$, $|Y| = 2p$, $|X| = d - p$ and $1 \leq |R| \leq p$. Then the corresponding counterparts of (4.6) involving such tuples give a basis for the homogeneous co-ordinate ring of $G_{d,n}$ related to the Plücker embedding of $G_{d,n}$ into $\mathbb{P}(\wedge^d \mathbb{C}^n)$; cf. [7].

5. One more representable class of balanced collections for arbitrary $p \geq q$ is obtained by slightly modifying the previous construction.

Fix a subset $Q \subseteq [p + 2..p + q]$ and form the collection $\mathcal{C} := \{A \in \binom{[p+q]}{p}: A \cap [p + 2..p + q] = Q\}$. We partition \mathcal{C} into two subcollections

$$\mathcal{A} := \{A \in \mathcal{C}: \Sigma(A) \text{ odd}\} \quad \text{and} \quad \mathcal{A}' := \{A \in \mathcal{C}: \Sigma(A) \text{ even}\}. \quad (4.7)$$

Lemma 4.2 *The pair $\mathcal{A}, \mathcal{A}'$ as in (4.7) is balanced.*

Proof Let $A \in \mathcal{C}$ and $M \in \mathcal{M}(A)$. Take the short arc $(i, i+1) \in M$ with i minimum. We assert that $i \leq p$. For otherwise any arc $i'j' \in M$ would satisfy $j' > p+1$ (by the argument as in the proof of Lemma 4.1), which is impossible since $|M| = q$ and $(p+q) - (p+1) < q$.

Thus, the set $A' := A \triangle \{i, i+1\}$ belongs to \mathcal{C} as well. Furthermore, A, A' belong to different collections among $\mathcal{A}, \mathcal{A}'$. Associating such $(A, M), (A', M)$ to each other, we obtain the desired bijection between $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A}')$. \blacksquare

This lemma gives rise to the corresponding class of sq-relations; we omit it here.

6. Our last illustration to the method concerns sq-relations analogous to ones yielding a Gröbner basis for the ideal $\ker(\psi)$ mentioned in the Introduction (cf. [10, Sec. 14.2]).

We identify a subset of $[n]$ with the sequence of its elements in the increasing order and consider the known partial order on $2^{[n]}$ in which for subsets $A = (a_1 < \dots < a_p)$ and $B = (b_1 < \dots < b_q)$, one puts $A \prec B$ if $p \geq q$ and $a_i \leq b_i$ for $i = 1, \dots, q$.

For p, q as before (i.e. $p \geq q$ and $p+q \leq n$), take a set $B = (b_1 < \dots < b_p) \in \binom{[p+q]}{p}$ incomparable with its complement $\overline{B} = (\overline{b}_1 < \dots < \overline{b}_q)$. Then there is $d \leq q$ such that $b_d > \overline{b}_d$ (usually one takes the smallest d with this property, but this is not important for us). Using this d , we partition each of B, \overline{B} into two subsets (where B^{left} or $\overline{B}^{\text{right}}$ may be empty):

$$\begin{aligned} B^{\text{left}} &:= \{b_1, \dots, b_{d-1}\}, & B^{\text{right}} &:= \{b_d, \dots, b_p\}, \\ \overline{B}^{\text{left}} &:= \{\overline{b}_1, \dots, \overline{b}_d\}, & \overline{B}^{\text{right}} &:= \{\overline{b}_{d+1}, \dots, \overline{b}_q\}, \end{aligned}$$

and form the set

$$C := \overline{B}^{\text{left}} \cup B^{\text{right}}.$$

By the choice of d , C begins with d black elements and ends with $p-d+1$ white elements (thinking of elements of B and \overline{B} as white and black, respectively). Introduce the following collection of p -element subsets of $[p+q]$:

$$\mathcal{B} := \{B^{\text{left}} \cup Z : Z \subset C, |Z| = p-d+1\}.$$

Then the complement \overline{A} of any member $A = B^{\text{left}} \cup Z$ of \mathcal{B} is the q -element set $(C - Z) \cup \overline{B}^{\text{right}}$. In other words, for each $A \in \mathcal{B}$, the pair (A, \overline{A}) is obtained from (B, \overline{B}) by swapping $\delta \leq \min\{d, p-d+1\}$ black and white elements in the parts $\overline{B}^{\text{left}}$ and B^{right} of C , respectively. We partition \mathcal{B} into two collections:

$$\mathcal{A} := \{A \in \mathcal{B} : \Sigma(A) \text{ odd}\} \quad \mathcal{A}' := \{A \in \mathcal{B} : \Sigma(A) \text{ even}\}. \quad (4.8)$$

In particular, the set B belongs to one of these collections (and is the greatest element in the poset (\mathcal{B}, \prec)).

Lemma 4.3 *The pair $\mathcal{A}, \mathcal{A}'$ as in (4.8) is balanced.*

Proof Consider $A \in \mathcal{B}$ and $M \in \mathcal{M}(B)$. The set C contains exactly d elements of \overline{A} , and the set $[p+q] - C$ contains $d-1$ elements of A (those forming B^{left}). Also $|M| = |\overline{A}| = q$.

Therefore, M contains at least one arc (i, j) with both ends i, j in C . We take such an arc (i, j) with j minimum and associate to (A, M) the configuration (A', M) with $A' := A \triangle \{i, j\}$. Then $A' \in \mathcal{B}$. Moreover, the fact that $j - i$ is odd (since the interval $[i..j]$ is partitioned into arcs in M) implies that $\Sigma(A) - \Sigma(A')$ is odd, whence A and A' belong to different collections among $\mathcal{A}, \mathcal{A}'$. Finally, by the canonical choice of (i, j) , the configuration associated to (B', M) is just (B, M) . \blacksquare

5 Necessity of the balancedness

In this section we prove the other direction in Theorem 3.1. In fact, a sharper property takes place, saying that for non-balanced $\mathcal{A}, \mathcal{A}'$, the corresponding quadratic relation is not valid in a very special case.

Proposition 5.1 *Let $\mathcal{A}, \mathcal{A}' \subseteq \binom{[p+q]}{p}$ be not balanced. Then (1.8) with these $\mathcal{A}, \mathcal{A}'$ is violated for some (G, w) already in case $\mathfrak{S} = \mathbb{R}$ and $n = p + q$ (and hence $X = \emptyset$ and $Y = [p+q]$). More precisely, there exist a planar network $G = (V, E)$ with $p+q$ sources and a weighting $w : V \rightarrow \mathbb{R}$ such that the flow-generated function $f = f_w$ on $2^{[p+q]}$ determined by w gives*

$$\sum_{A \in \mathcal{A}} f(A)f(\overline{A}) \neq \sum_{A \in \mathcal{A}'} f(A)f(\overline{A}). \quad (5.1)$$

Proof Since $\mathcal{A}, \mathcal{A}'$ are not balanced, there exists a nested matching M in $[p+q]$ (with $|M| = q$) such that

$$|\mathcal{A}_M| \neq |\mathcal{A}'_M|, \quad (5.2)$$

where \mathcal{A}_M denotes the set of members $A \in \mathcal{A}$ having M as a feasible matching: $M \in \mathcal{M}(A)$, and similarly for \mathcal{A}' .

We fix such an M , and our aim is to construct a planar network $G = (V, E)$ with $p+q$ sources that satisfies the following properties:

- (P1) for each $A \in \binom{[p+q]}{p}$ with $M \in \mathcal{M}(A)$, G has a unique A -flow and a unique \overline{A} -flow, i.e. $|\Phi_A| = |\Phi_{\overline{A}}| = 1$;
- (P2) if $A \in \binom{[p+q]}{p}$ and $M \notin \mathcal{M}(A)$, then at least one of Φ_A and $\Phi_{\overline{A}}$ is empty.

Once we are given such a G , assign $w(v) := 1$ for all $v \in V$. In view of (P1) and (P2), for $f = f_w$ and $A \in \binom{[p+q]}{p}$, we have $f(A) = f(\overline{A}) = 1$ if $M \in \mathcal{M}(A)$, and $f(A) = f(\overline{A}) = 0$ otherwise (equivalently, the term $f(A)f(\overline{A})$ vanishes in the latter case). This implies $\sum_{A \in \mathcal{A}} f(A)f(\overline{A}) = |\mathcal{A}_M|$ and $\sum_{A \in \mathcal{A}'} f(A)f(\overline{A}) = |\mathcal{A}'_M|$, and now the required inequality (5.1) follows from (5.2).

We first construct the desired network G in case $p = q$. This network is embedded in the upper half-plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$ and its sources s_i are identified with the points $(i, 0)$, $i = 1, \dots, 2p$. The remaining part of G is designed as follows.

If two elements (arcs) $\pi, \pi' \in M$ obey $[\pi] \supset [\pi']$ and if there is no $\pi'' \in M$ such that $[\pi] \supset [\pi''] \supset [\pi']$, we say that π is the *immediate predecessor* of π' and that π' is an *immediate successor* of π . An arc in M having no predecessor is called *maximal* (so the intervals of maximal arcs are pairwise disjoint and their union is $[2p]$).

Consider an arc $\pi = (i, j)$ and let $\Delta = \Delta(\pi) := (j - i + 1)/2$. We represent π by graph C_π consisting of $2\Delta + 1$ vertices and 2Δ edges whose union forms the half-circumference ξ_π lying in the upper half-plane, connecting the points s_i and s_j and having the center at the point $(\frac{i+j}{2}, 0)$. More precisely, the vertices of C_π lie on ξ_π and are labeled as $s_i = v_0, u_1, v_1, u_2, \dots, v_{\Delta-1}, u_\Delta, v_\Delta = s_j$, in this order from left to right, and the directed edges (formed by the pieces of ξ_π between consecutive vertices) correspond to the pairs $(v_{\ell-1}, u_\ell)$ and (v_ℓ, u_ℓ) for $\ell = 1, \dots, \Delta$. To indicate the arc in M generating these vertices, we also write u_ℓ^π for u_ℓ , and v_ℓ^π for v_ℓ .

Let G' be the (disjoint) union of graphs C_π , $\pi \in M$. The desired network G is obtained from G' by drawing additional edges connecting subgraphs C_π and $C_{\pi'}$ for each non-maximal π' and its immediate predecessor π . More precisely, let $\pi = (i, j) \in M$ and let $\pi_\alpha = (i_\alpha, j_\alpha)$, $\alpha = 1, \dots, k$, be the immediate successors of π (possibly $k = 0$); we assume that the successors are indexed from left to right, i.e. $i_\alpha = j_{\alpha-1} + 1$ (then $i_1 = i + 1$ and $j_k = j - 1$). Observe that

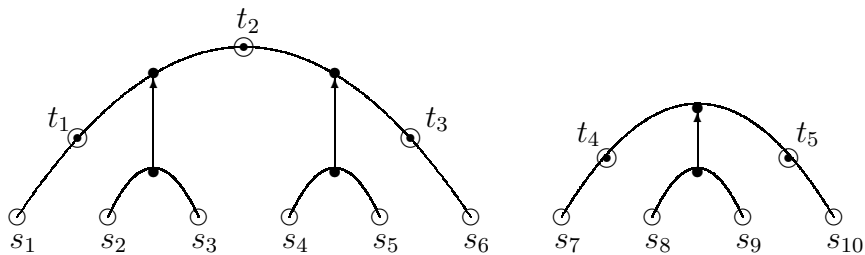
$$\sum_{\alpha=1}^k \Delta(\pi_\alpha) = \Delta(\pi) - 1.$$

So the total number of u -vertices in the subgraphs C_{π_α} ($\alpha = 1, \dots, k$) is equal to the number of v -vertices in C_π different from the endvertices $v_0^\pi = s_i$ and $v_{\Delta(\pi)}^\pi = s_j$. Moving from left to right (and preserving the planarity), we connect these vertices by $\Delta(\pi) - 1$ edges directed from u - to v -vertices. Formally, for $\alpha = 1, \dots, k$ and $\ell = 1, \dots, \Delta(\pi_\alpha)$, we draw edge $e_{\pi_\alpha, \ell}$ from $u_\ell^{\pi_\alpha}$ to $v_{(i_\alpha - i - 1)/2 + \ell}^\pi$.

Finally, let π_1, \dots, π_h be the maximal arcs in M , in this order from left to right. Then $\Delta(\pi_1) + \dots + \Delta(\pi_h) = p$. The concatenation (from left to right) of the sequences of u -vertices in $C_{\pi_1}, \dots, C_{\pi_h}$ is

$$u_1^{\pi_1}, \dots, u_{\Delta(\pi_1)}^{\pi_1}, u_1^{\pi_2}, \dots, u_{\Delta(\pi_2)}^{\pi_2}, \dots, u_1^{\pi_h}, \dots, u_{\Delta(\pi_h)}^{\pi_h}.$$

We assign these p vertices to be the sinks of G , denoted as t_1, \dots, t_p , respectively. This completes the construction of G with specified sources and sinks. The picture illustrates an example of G ; here $p = 5$ and $M = \{16, 23, 45, (7, 10), 89\}$.



Remark 3. The constructed G is a forest whose connected components (trees) correspond to maximal arcs in M . One can make it connected, e.g., by adding extra nodes z_i and edges $(z_i, s_i), (z_i, s_{i+1})$ for $i = 1, \dots, 2p - 1$. Then the sources and sinks become lying in the boundary of the outer face and go there in the order $s_{2p}, \dots, s_1, t_1, \dots, t_p$. In fact, it does not matter that our network has p (rather than $2p$) sinks; however, to be consistent with settings in Section 1, we can slightly modify the network so as to add sinks t_{p+1}, \dots, t_{2p} in a due way.

We assert that G satisfies properties (P1) and (P2) for the given M . To show (P1), we use induction on $|M|$. Consider $A \in \binom{[2p]}{p}$ such that $M \in \mathcal{M}(A)$. Let $\pi = (1, r)$ be the first maximal arc in M , and let $M' := M - \{\pi\}$. By our construction, the corresponding network G' for M' is obtained from G by removing the subgraph C_π (and the edges connecting it with the rest of G). It has $p - 1$ sinks t'_1, \dots, t'_{p-1} , of which the last $p - \Delta(\pi)$ sinks are sinks of G (namely, $t'_\ell = t_{\ell+1}$ for $\ell = p - \Delta(\pi), \dots, p - 1$), and the first $\Delta(\pi) - 1$ sinks $t'_1, \dots, t'_{\Delta(\pi)-1}$ are the u -vertices in the subgraphs $C_{\pi'}$ generated by the immediate successors π' of π in M .

The $(p - 1)$ -element sets $A' := A - \{1, r\}$ and $\tilde{A}' := \bar{A} - \{1, r\}$ give a partition of the set $[2p] - \{1, r\}$ (whose elements can be renumbered as $1', 2', \dots$ if wished), and both have M' as a feasible matching. By induction G' has a unique A' -flow F' and a unique \tilde{A}' -flow \tilde{F}' . Observe that G has a unique collection \mathcal{P} of $\Delta(\pi)$ pairwise disjoint directed paths going, respectively, from the vertices $s_1, t'_1, \dots, t'_{\Delta(\pi)-1}$ to the vertices $t_1, \dots, t_{\Delta(\pi)}$ (namely, the path from s_1 to t_1 consists of one edge, and the path from t'_ℓ to $t_{\ell+1}$ goes through the vertex v_ℓ). Similarly, G has a unique collection \mathcal{Q} of pairwise disjoint directed paths going, respectively, from $t'_1, \dots, t'_{\Delta(\pi)-1}, s_r$ to $t_1, \dots, t_{\Delta(\pi)}$. Now the (unique) A -flow \mathcal{F} and \bar{A} -flow $\bar{\mathcal{F}}$, as required in (P1) for G , are obtained by combining F', \tilde{F}' with \mathcal{P}, \mathcal{Q} in a natural way. (In case $A = A' \cup \{1\}$, \mathcal{F} is formed from F', \mathcal{P} , and $\bar{\mathcal{F}}$ from \tilde{F}', \mathcal{Q} . In case $A = A' \cup \{r\}$, \mathcal{F} is formed from F', \mathcal{Q} , and $\bar{\mathcal{F}}$ from \tilde{F}', \mathcal{P} .)

Next we show (P2). Let $A \in \binom{[2p]}{p}$ and $M \notin \mathcal{M}(A)$. Then there exists $\pi = (i, j) \in M$ such that one of A, \bar{A} contains both i, j . Choose π with $j - i$ minimum under this property. Assume that $i, j \in A$. Since each arc $\pi' \in M$ with $[\pi'] \subset [\pi]$ has exactly one end in A and since $i, j \in A$, we observe that the interval $[\pi]$ contains $\Delta(\pi) + 1$ elements of A . On the other hand, the number of u -vertices in the subgraph C_π is only $\Delta(\pi)$ (and removing these vertices from G disconnects the sources s_k with $i \leq k \leq j$ from the sinks). Hence no A -flow in G can exist. Similarly, when $i, j \in \bar{A}$, the network G has no \bar{A} -flow. This yields (P2).

It remains to consider the case $p > q$. We reduce it to the previous case as follows. Given a nested matching M (of size q) in $[p + q]$ satisfying (5.2), let $i_1 < \dots < i_{p-q}$ be the sequence of free elements for M . We involve these elements in $p - q$ new arcs $\pi_\ell = (i_\ell, 2p - \ell + 1)$, $\ell = 1, \dots, p - q$, which are added to M . One can see that the resulting arc set \widehat{M} is a correct nested matching of size p in $[2p]$. Accordingly, for each $A \in \binom{[p+q]}{p}$, the partition (A, \bar{A}) of $[p + q]$ is associated with the partition (A, \hat{A}) of $[2p]$, where $\hat{A} := \bar{A} \cup \{p + q + 1, \dots, 2p\}$. Let G be the network constructed as above for the matching \widehat{M} in $[2p]$, and let s_1, \dots, s_{2p} and t_1, \dots, t_p be the sequences of sources

and sinks in it, respectively. It is not difficult to check that G satisfies the required properties (P1) and (P2) for the initial matching M as well. (If wished, one can remove the sources s_{p+q+1}, \dots, s_{2p} from G .)

This completes the proof of the proposition. ■

6 The standard basis and Laurent phenomenon

In this section we assume that \mathfrak{S} is a commutative semiring with division, i.e. \mathfrak{S} contains $\underline{1}$ and the operation \odot is invertible (in other words, (\mathfrak{S}, \odot) is an abelian group). Two important special cases, mentioned in the Introduction, are: the set $\mathbb{R}_{>}$ of positive reals; the tropicalization $\mathfrak{L}^{\text{trop}}$ of a totally ordered abelian group \mathfrak{L} , in particular, the set \mathbb{R}_{\max} of reals with operations $\oplus = \max$ and $\odot = +$. In these special cases the corresponding sets \mathbf{FG} of flow-generated functions on $2^{[n]}$ (namely, $\mathbf{FG}_{\mathbf{n}}(\mathbb{R}_{>})$ and $\mathbf{FG}_{\mathbf{n}}(\mathfrak{L}^{\text{trop}})$) possess the following nice properties: (i) all these functions f can be generated by flows in one planar network, namely, in the half-grid $\Gamma = \Gamma_n$; (ii) \mathbf{FG} has as a basis the set \mathcal{I}_n of intervals in $[n]$ (called the *standard* basis for \mathbf{FG}), and (iii) the values of f are expressed as (algebraic or tropical) Laurent polynomials in its values on \mathcal{I}_n . These facts are discussed in [3] (mostly for \mathbb{R}_{\max}) and in [2, 5] (concerning (iii)); in essence, the arguments can be directly extended to an arbitrary \mathfrak{S} as above. Below we give a brief outline (which is sufficient to restore the details with help of [3]).

An important feature of this $\Gamma = (V, E)$ is that for any nonempty interval $I = [q..r]$ in $[n]$, there exists exactly one feasible flow ϕ_I from S_I to the sinks $t_1, \dots, t_{|I|}$; namely, ϕ_I goes through the vertices (i, j) occurring in the rectangle $[r] \times [r - q + 1]$ (more precisely, satisfying $i \leq r$, $j \leq r - q + 1$ and $i \geq j$). Therefore, given a weighting $w : V \rightarrow \mathfrak{S}$, the values of $f = f_w$ on the nonempty intervals $[q..r]$ are viewed as

$$f[q..r] = \bigodot_{j \leq i \leq r, 1 \leq j \leq r-q+1} w(i, j). \quad (6.1)$$

Note that the number $\frac{n(n+1)}{2}$ of vertices of Γ is equal to the number of nonempty intervals in $[n]$ and the system (6.1) is non-degenerate. So, using division in \mathfrak{S} , denoted as $/$, we can in turn express the weights of vertices via the values of f on the intervals. This is computed as

$$w(i, j) = \begin{cases} (f(I_{i,j}) \odot f(I_{i-1,j-1})) / (f(I_{i-1,j}) \odot f(I_{i,j-1})) & \text{for } i > j, \\ f(I_{i,j}) / f(I_{i,j-1}) & \text{for } i = j, \end{cases} \quad (6.2)$$

denoting by $I_{i',j'}$ the interval $[(i' - j' + 1)..i']$ and letting $f(I_{i',0}) := \underline{1}$.

Thus, the correspondence $w \mapsto f_w$ gives a bijection between the set of weightings $w : V \rightarrow \mathfrak{S}$ and $\mathfrak{S}^{\mathcal{I}_n^0}$, where \mathcal{I}_n^0 is the set of nonempty intervals in $[n]$. By definition (1.7), the value of $f = f_w$ on any nonempty subset $A \subseteq [n]$ is represented by a “polynomial” in variables $w(v)$, $v \in V$, namely, by a \oplus -sum of products $\odot(w(v) : v \in V')$ for some subsets $V' \subseteq V$. Substituting into this polynomial the corresponding terms from (6.2), we obtain an expression of the form

$$f(A) = \oplus (\mathcal{P}_k : k = 1, \dots, N),$$

where each \mathcal{P}_k is a “monomial” $\odot(f(I)^{\sigma_k(I)} : I \in \mathcal{I}_n^0)$ with integer (possibly negative) degrees $\sigma_k(I)$. This means that $f(A)$ is a Laurent polynomial (regarding addition \oplus and multiplication \odot) in variables $f(I)$, $I \in \mathcal{I}_n^0$.

Remark 4. Analyzing possible flows in Γ , one can show that the degrees $\sigma_k(I)$ are bounded and, moreover, belong to $\{-1, 0, 1, 2\}$. This is proved in [3] for the tropical case and can be straightforwardly extended to an arbitrary commutative semiring \mathfrak{S} with division. (In [12] similar bounds are established for Laurent polynomials arising in the octahedron recurrence.)

Finally, a simple fact (see [3]) is that any function $f : 2^{[n]} \rightarrow \mathbb{R}$ obeying TP3-relation (1.6) is determined by its values on \mathcal{I}_n . The proof of this fact is directly extended to \mathfrak{S} in question. (A hint: if $S \subseteq [n]$ is not an interval, define $i := \min(S)$, $k := \max(S)$, $X := S - \{i, k\}$, and let j be an element in $[i..k] - S$. Then for a function f on $2^{[n]}$ obeying SP3-relation (4.1), the value $f(S)$ is expressed via the values $f(S')$ on five sets $S' = Xi, Xj, Xk, Xij, Xjk$. Since $\max(S') - \min(S') < \max(S) - \min(S)$, we can apply induction on $\max(S) - \min(S)$.) This fact together with reasonings above implies that \mathcal{I}_n is a basis for the functions in $\mathbf{FG}_n(\mathfrak{S})$ and that all these functions are generated by flows in Γ (so they are bijective to weightings $w : V \rightarrow \mathfrak{S}$, up to their values on \emptyset , and possess the Laurentness property as above).

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